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Finite-genus solutions for the Ablowitz–Ladik hierarchy

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Abstract. The question of constructing the finite-genus quasiperiodic solutions for the Ablowitz–Ladik hierarchy (ALH) is considered by establishing relations between the ALH and Fay's identity for the θ-functions. It is shown that using a limiting procedure one can derive from the latter an infinite number of differential identities, which can be arranged as an infinite set of differential-difference equations coinciding with the equations of the ALH, and that the original Fay's identity can be rewritten in a form similar to the functional equations representing the ALH which have been derived in the previous works of the author. This provides an algorithm for obtaining some class of quasiperiodic solutions for the ALH, which can be viewed as an alternative to the inverse scattering transform or the algebro-geometrical approach.

1. Introduction

The problem of constructing quasiperiodic solutions (QPS) is one of the most challenging problems of the theory of integrable systems, and many mathematicians and physicists have spent much effort to obtain the QPS for almost all equations that are known to be integrable. The Ablowitz–Ladik hierarchy (ALH), which has been introduced in [1], is no exception. So, for example, one should mention the works by Bogolyubov and co-workers [2, 3] and Ahmad and Chowdhury [4, 5] devoted to the discrete nonlinear Schrödinger equation (DNLSE) and the discrete modified Korteveg–de Vries equation (DMKdV), which are the best studied equations of the ALH. There authors were studying this problem within the framework of the inverse scattering transform (IST). Another, the so-called algebraic-geometrical, approach has been used by Miller *et al* [6] who considered the complex version of the DNLSE and obtained the Baker–Akhiezer function and QPS corresponding to finite-genus Riemann surfaces. This work provides an almost exhaustive solution of the problem of the finite-genus QPS, but its results need some further simplification to be useful for practical purposes, especially if one wants to extend them to the higher equations of the ALH and in this work I will try to avoid algebro-geometrical language, and will use a more direct (and simpler) strategy. As has been established in [2–6], each finite-genus QPS of the DNLSE can be presented as a quotient of the θ-functions of some arguments multiplied by an exponent of some phase, all of them being some *linear* functions of the coordinates (the same is true in the cases of the DMKdV as well as all other equations of the hierarchy). Thus, since we know the structure of the solutions, all we have to do to derive them is to calculate some number of constant parameters. So, it is desirable to develop some method, which will enable us to obtain these constants (and hence solutions) straightforwardly, not using technique (sometimes rather complicated) of the theory of functions and differentials on hyperelliptic Riemann surfaces. It turns out that this can be done. Moreover, this can be done not only for the DNLSE or DMKdV but, in principle,

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for all equations of the hierarchy simultaneously. Namely, this is the main question of this paper. The key point is that the θ-functions of the finite-genus Riemann surfaces (of which the finite-genus QPS are built-up) satisfy some algebraic relation, the so-called Fay's trisecant formula [7, 8], which can be used to obtain an infinite number of differential identities, which, as will be shown below, are closely related to the ALH, and can be used to obtain the QPS we are looking for. Such an approach also demonstrates some new, to my knowledge, feature of the ALH (and this was one of the main motivations to write this paper): the equations of the ALH naturally appear when flows over Riemann surfaces are considered (I will return to this question below).

The paper is organized as follows. After presenting some basic facts on the ALH (section 2) I will discuss Fay's formula and its differential consequences (section 3). These results will be used to obtain the finite-genus QPS for the ALH (section 4).

2. Ablowitz–Ladik hierarchy

The ALH is an infinite set of integrable differential-difference equations, which was introduced in [1]. All equations of the ALH can be presented as the compatibility condition for the linear system

$$\Psi_{n+1} = U_n \Psi_n \tag{2.1}$$

$$\frac{\partial}{\partial z_j} \Psi_n = V_n^{(j)} \Psi_n \qquad j = \pm 1, \pm 2, \ldots \tag{2.2}$$

where Ψ_n is a 2-column, U_n and V_n are 2×2 matrices with U_n being given by

$$U_n = U_n(\lambda) = \begin{pmatrix} \lambda & r_n \\ q_n & \lambda^{-1} \end{pmatrix} \tag{2.3}$$

(here λ is the auxiliary (spectral) constant parameter) and the matrices $V_n^{(j)}$ are polynomials in λ, λ^{-1}. The ALH can be split in a natural way into two subsystems (subhierarchies). One of them corresponds to the case when $V_n^{(j)}$, $j = 1, 2, \ldots$, are jth-order polynomials in λ^{-1} ('positive' subhierarchy). Its simplest equations are

$$\frac{\partial q_n}{\partial z_1} = -\mathrm{i} p_n q_{n+1} \tag{2.4}$$

$$\frac{\partial r_n}{\partial z_1} = \mathrm{i} p_n r_{n-1} \tag{2.5}$$

where

$$p_n = 1 - q_n r_n. \tag{2.6}$$

The 'negative' subhierarchy is build up of the V-matrices being polynomials in λ. Its simplest equations are

$$\frac{\partial q_n}{\partial \bar{z}_1} = -\mathrm{i} p_n q_{n-1} \tag{2.7}$$

$$\frac{\partial r_n}{\partial \bar{z}_1} = \mathrm{i} p_n r_{n+1}. \tag{2.8}$$

(I use the notation $\bar{z}_j = z_{-j}$, $j = 1, 2, \ldots$).

It has been shown in [10] that the ALH can be presented in the form of the functional-difference equations:

$$q_n(z, \bar{z}) - q_n(z - \mathrm{i}[\xi], \bar{z}) = \xi [1 - q_n(z, \bar{z}) r_n(z - \mathrm{i}[\xi], \bar{z})] q_{n+1}(z, \bar{z}) \tag{2.9}$$

$$r_n(z, \bar{z}) - r_n(z + \mathrm{i}[\xi], \bar{z}) = \xi [1 - q_n(z + \mathrm{i}[\xi], \bar{z}) r_n(z, \bar{z})] r_{n-1}(z, \bar{z}) \tag{2.10}$$

for the 'positive' subhierarchy, and

$$q_n(z, \bar{z}) - q_n(z, \bar{z} - \mathrm{i}[\xi^{-1}]) = \xi^{-1}[1 - q_n(z, \bar{z})r_n(z, \bar{z} - \mathrm{i}[\xi^{-1}])]q_{n-1}(z, \bar{z}) \tag{2.11}$$

$$r_n(z, \bar{z}) - r_n(z, \bar{z} + \mathrm{i}[\xi^{-1}]) = \xi^{-1}[1 - q_n(z, \bar{z} + \mathrm{i}[\xi^{-1}])r_n(z, \bar{z})]r_{n+1}(\bar{z}, z) \tag{2.12}$$

for the 'negative' one. Here the designations

$$f(z, \bar{z}) = f(z_1, z_2, z_3, \ldots, \bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots) \tag{2.13}$$

and

$$f(z \pm \mathrm{i}[\xi], \bar{z}) = f\big(z_1 \pm \mathrm{i}\xi, z_2 \pm \mathrm{i}\xi^2/2, z_3 \pm \mathrm{i}\xi^3/3, \ldots, \bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots\big) \tag{2.14}$$

$$f(z, \bar{z} \pm \mathrm{i}[\xi^{-1}]) = f\big(z_1, z_2, z_3, \ldots, \bar{z}_1 \pm \mathrm{i}\xi^{-1}, \bar{z}_2 \pm \mathrm{i}\xi^{-2}/2, \bar{z}_3 \pm \mathrm{i}\xi^{-3}/3, \ldots\big) \tag{2.15}$$

are used. Expanding equations (2.9) and (2.10) in power series in ξ one can obtain all equations of the 'positive' subhierarchy. Analogously, expanding equations (2.11) and (2.12) in power series in ξ^{-1} one can obtain all equations of the 'negative' one.

In what follows I will also use the tau-functions of the ALH, σ_n, ρ_n and τ_n, which are defined by

$$q_n = \frac{\sigma_n}{\tau_n} \qquad r_n = \frac{\rho_n}{\tau_n} \tag{2.16}$$

and

$$\tau_{n-1}\tau_{n+1} = \tau_n^2 - \sigma_n\rho_n. \tag{2.17}$$

The functional representation of the ALH in terms of the tau-functions can be written as

$$\tau_n(z)\,\sigma_n(z + \mathrm{i}[\xi]) - \sigma_n(z)\,\tau_n(z + \mathrm{i}[\xi]) = \xi\;\tau_{n-1}(z)\,\sigma_{n+1}(z + \mathrm{i}[\xi]) \tag{2.18}$$

$$\rho_n(z)\,\tau_n(z + \mathrm{i}[\xi]) - \tau_n(z)\,\rho_n(z + \mathrm{i}[\xi]) = \xi\;\rho_{n-1}(z)\,\tau_{n+1}(z + \mathrm{i}[\xi]) \tag{2.19}$$

$$\tau_n(z)\,\tau_n(z + \mathrm{i}[\xi]) - \rho_n(z)\,\sigma_n(z + \mathrm{i}[\xi]) = \tau_{n-1}(z)\,\tau_{n+1}(z + \mathrm{i}[\xi]) \tag{2.20}$$

(where the dependence on $\bar{z}_j$ is omitted) and

$$\tau_n(\bar{z})\,\sigma_n(\bar{z} + \mathrm{i}[\xi^{-1}]) - \sigma_n(\bar{z})\,\tau_n(\bar{z} + \mathrm{i}[\xi^{-1}]) = \xi^{-1}\;\tau_{n+1}(\bar{z})\,\sigma_{n-1}(\bar{z} + \mathrm{i}[\xi^{-1}]) \tag{2.21}$$

$$\rho_n(\bar{z})\,\tau_n(\bar{z} + \mathrm{i}[\xi^{-1}]) - \tau_n(\bar{z})\,\rho_n(\bar{z} + \mathrm{i}[\xi^{-1}]) = \xi^{-1}\;\rho_{n+1}(\bar{z})\,\tau_{n-1}(\bar{z} + \mathrm{i}[\xi^{-1}]) \tag{2.22}$$

$$\tau_n(\bar{z})\,\tau_n(\bar{z} + \mathrm{i}[\xi^{-1}]) - \rho_n(\bar{z})\,\sigma_n(\bar{z} + \mathrm{i}[\xi^{-1}]) = \tau_{n+1}(\bar{z})\,\tau_{n-1}(\bar{z} + \mathrm{i}[\xi^{-1}]) \tag{2.23}$$

(where the dependence on z_j is omitted) [9, 10].

The key idea of the present work is to establish the relation between these equations and Fay's famous identity for the θ-functions, which can be used to derive the finite-gap QPS of the ALH.

3. Fay's identity

In this paper we will deal with the compact Riemann surface X of the genus g corresponding to the hyperelliptic curve

$$s^2 = \mathcal{P}_{2g+2}(\xi) \tag{3.1}$$

where $\mathcal{P}_{2g+2}(\xi)$ is a polynomial without multiple roots of degree $2g + 2$. In the framework of the IST such curves appear in the analysis of the scattering problem (2.1). For example, in the case of the periodic conditions

$$q_{n+g+1} = q_n \qquad r_{n+g+1} = r_n \tag{3.2}$$

the polynomial $\mathcal{P}_{2g+2}(\xi)$ is defined by

$$\mathcal{P}_{2g+2}(\lambda^2) = \lambda^{2(g+1)}\left\{[\mathrm{tr}\, T_n(\lambda)]^2 - 4\det T_n(\lambda)\right\} \tag{3.3}$$

where $T_n(\lambda)$ is the transfer matrix of the scattering problem (2.1),

$$T_n(\lambda) = U_{n+g}(\lambda)\ldots U_n(\lambda) \tag{3.4}$$

(it can be shown straightforwardly that the right-hand side of (3.3) under the restriction (3.2) does not depend on the index n). Topologically, X is a sphere with g handles. One can choose a set of $2g$ closed contours (cycles) $\{a_i, b_i\}_{i=1,\ldots,g}$ with the intersection indices

$$a_i \circ a_j = b_i \circ b_j = 0 \qquad a_i \circ b_j = \delta_{ij} \qquad i, j = 1, \ldots, g \tag{3.5}$$

and find g independent holomorphic differentials, say ones given locally by

$$\widetilde{\omega}_k = \frac{\xi^{k-1}\,\mathrm{d}\xi}{\sqrt{\mathcal{P}_{2g+2}(\xi)}} \qquad k = 1, \ldots, g \tag{3.6}$$

which can be used to construct the canonical basis of the holomorphic 1-forms

$$\omega_k = \sum_{l=1}^{g} C_{k,l}\widetilde{\omega}_l \tag{3.7}$$

where ω_k satisfy the normalization conditions

$$\oint_{a_i} \omega_k = \delta_{ik}. \tag{3.8}$$

Then, the matrix of the b-periods,

$$\Omega_{ik} = \oint_{b_i} \omega_k \tag{3.9}$$

determines the so-called period lattice, $L_\Omega = \{\boldsymbol{m} + \Omega\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^g\}$, the Jacobian of this surface $\mathrm{Jac}(X) = \mathbb{C}^g/L_\Omega$ ($2g$ torus) and Abel mapping $X \to \mathrm{Jac}(X)$,

$$P \to \int_{P_0}^{P} \boldsymbol{\omega} \tag{3.10}$$

where $\boldsymbol{\omega}$ is the g-vector of the 1-forms, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_g)^T$ and P_0 is some fixed point of X.

A central object of the theory of the compact Riemann surfaces is the θ-function, $\theta(\boldsymbol{\zeta}) = \theta(\boldsymbol{\zeta}, \Omega)$,

$$\theta(\boldsymbol{\zeta}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\{\pi \mathrm{i}\boldsymbol{n}\Omega\boldsymbol{n} + 2\pi\mathrm{i}\boldsymbol{n}\boldsymbol{\zeta}\} \tag{3.11}$$

which is a quasiperiodic function on $\mathbb{C}^g$

$$\theta(\boldsymbol{\zeta} + \boldsymbol{n}) = \theta(\boldsymbol{\zeta}) \tag{3.12}$$

$$\theta(\boldsymbol{\zeta} + \Omega\boldsymbol{n}) = \exp\{-\pi\mathrm{i}\boldsymbol{n}\Omega\boldsymbol{n} - 2\pi\mathrm{i}\boldsymbol{n}\boldsymbol{\zeta}\}\theta(\boldsymbol{\zeta}) \tag{3.13}$$

for $\boldsymbol{n} \in \mathbb{Z}^g$.

To simplify the following formulae I will use the designations

$$\theta_B^A(\boldsymbol{\zeta}) = \theta\left(\boldsymbol{\zeta} + \int_B^A \boldsymbol{\omega}\right) \tag{3.14}$$

and

$$\hat{\theta}_B^A = \theta[\boldsymbol{\delta}]\left(\int_B^A \boldsymbol{\omega}\right). \tag{3.15}$$

Here $\theta[\boldsymbol{c}](\boldsymbol{\zeta})$ is the so-called θ-function with characteristics,

$$\theta[\boldsymbol{c}](\boldsymbol{\zeta}) = \exp\{\pi \mathrm{i}\boldsymbol{a}\Omega\boldsymbol{a} + 2\pi\mathrm{i}\boldsymbol{a}(\boldsymbol{\zeta}+\boldsymbol{b})\}\theta(\boldsymbol{\zeta}+\Omega\boldsymbol{a}+\boldsymbol{b}) \qquad \boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \tag{3.16}$$

and $\boldsymbol{\delta} = (\boldsymbol{\delta}', \boldsymbol{\delta}'') \in \frac{1}{2}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}$ is a non-singular odd characteristic,

$$\theta[\boldsymbol{\delta}](\mathbf{0}) = 0 \qquad \mathrm{grad}_{\zeta}\,\theta[\boldsymbol{\delta}](\mathbf{0}) \neq \mathbf{0}. \tag{3.17}$$

Function (3.15) is closely related to the prime form [7, 8],

$$E(P, Q) = \frac{\hat{\theta}_Q^P}{\sqrt{\chi(P)}\sqrt{\chi(Q)}} \tag{3.18}$$

where χ is given by

$$\chi(P) = \sum_{i=1}^{g}\left(\frac{\partial}{\partial\zeta_i}\,\theta[\boldsymbol{\delta}]\right)(\mathbf{0})\omega_i(P). \tag{3.19}$$

The prime form $E(P, Q)$ is skew-symmetric, $E(P, Q) = -E(Q, P)$, has a first-order zero along the diagonal $P = Q$ and is otherwise non-zero. Analogously,

$$\hat{\theta}_Q^P = -\hat{\theta}_P^Q \qquad \hat{\theta}_P^P = 0. \tag{3.20}$$

One of the most interesting results of the theory of the θ-functions is the following identity for the θ-functions associated with the finite-genus Riemann surfaces, Fay's identity:

$$\hat{\theta}_{P_3}^{P_1}\,\hat{\theta}_{P_2}^{P_4}\,\theta_{P_2}^{P_1}(\boldsymbol{\zeta})\,\theta_{P_3}^{P_4}(\boldsymbol{\zeta}) - \hat{\theta}_{P_2}^{P_1}\,\hat{\theta}_{P_3}^{P_4}\,\theta_{P_3}^{P_1}(\boldsymbol{\zeta})\,\theta_{P_2}^{P_4}(\boldsymbol{\zeta}) = \hat{\theta}_{P_3}^{P_2}\,\hat{\theta}_{P_1}^{P_4}\,\theta(\boldsymbol{\zeta})\,\theta_{P_2P_3}^{P_1P_4}(\boldsymbol{\zeta}) \tag{3.21}$$

(here $P_1, \ldots, P_4$ are arbitrary points of X) and namely this formula will be the basis of the following consideration.

4. Quasiperiodic solutions

It is already known that in the quasiperiodic case the tau-functions of the ALH are (up to some simple factors) the θ-functions of different arguments and I am now going to present Fay's identity and some of its corollaries in a form similar to (2.18)–(2.20) and (2.21)–(2.23), which will enable us to obtain the finite-gap solutions of these functional equations, i.e. to obtain the finite-gap solutions of the ALH.

Hereafter I will use the letters A, B, C and D for the points of the Riemann surface, which correspond to the points 0 and ∞ of the complex plane,

$$A = \infty_+ \qquad D = \infty_- \qquad B = 0_- \qquad C = 0_+. \tag{4.1}$$

Since A, D and B, C are poles and zeros of the meromorphic function $\pi(P)$, which is a projection of X onto the extended complex plane $\mathbb{P}^1$ sending a point $P = (s, \xi)$ into ξ, they satisfy, according to Abel's theorem, the condition $\int_{BC}^{AD}\boldsymbol{\omega} \in L_\Omega$. The integration paths in (3.10) can be chosen in such a way that

$$\int_{BC}^{AD}\boldsymbol{\omega} = \mathbf{0} \tag{4.2}$$

(here zero stands for $\mathbf{0}$ from $\mathbb{C}^g$, not from $\mathrm{Jac}(X)$) and in what follows I will accept (4.2) as true.

Now I am going to use (3.21) thinking of three points from (P_1, P_2, P_3, P_4) as constant (I will choose them from the set (A, B, C, D)) and the fourth one (I will denote it by P) as variable. Setting $(P_1, P_2, P_3, P_4) = (A, B, C, P)$ one can rewrite (3.21) as

$$\hat\theta_C^B\,\hat\theta_A^P\,\theta(\zeta)\,\theta_D^P(\zeta) + \hat\theta_B^A\,\hat\theta_C^P\,\theta_C^A(\zeta)\,\theta_B^P(\zeta) = \hat\theta_C^A\,\hat\theta_B^P\,\theta_B^A(\zeta)\,\theta_C^P(\zeta). \tag{4.3}$$

This formula is the quasiperiodic analogue of (2.18). Shifting the arguments of the θ-functions, $\zeta \to \zeta + \int_A^C \omega$, one can obtain the equation, which will be transformed below to (2.19):

$$\hat\theta_C^B\,\hat\theta_A^P\,\theta_A^C(\zeta)\,\theta_B^P(\zeta) + \hat\theta_B^A\,\hat\theta_C^P\,\theta(\zeta)\,\theta_{AB}^{CP}(\zeta) = \hat\theta_C^A\,\hat\theta_B^P\,\theta_B^C(\zeta)\,\theta_A^P(\zeta). \tag{4.4}$$

At last, replacing in (3.21) (P_1, P_2, P_3, P_4) with (A, P, C, D) using (4.2) and making the shift $\zeta \to \zeta + \int_A^C \omega$ one can write the identity

$$\hat\theta_D^A\,\hat\theta_C^P\,\theta(\zeta)\,\theta_B^P(\zeta) - \hat\theta_B^A\,\hat\theta_A^P\,\theta_A^C(\zeta)\,\theta_D^P(\zeta) = \hat\theta_C^A\,\hat\theta_D^P\,\theta_B^A(\zeta)\,\theta_A^P(\zeta) \tag{4.5}$$

which is a quasiperiodic analogue of (2.20).

Our first goal is to present equations (4.3)–(4.5) in the bilinear form. To this end I will first shift the arguments of the θ-functions: $\zeta \to \zeta_n$,

$$\zeta_n = \zeta + n\int_A^B \omega. \tag{4.6}$$

Next, I will introduce the functions $\sigma_n(P)$, $\rho_n(P)$ and $\tau_n(P)$,

$$\tau_n(P) = \alpha_n(P)\,\theta_B^P(\zeta_n) \tag{4.7}$$

$$\sigma_n(P) = \beta_n(P)\,\theta_D^P(\zeta_n) \tag{4.8}$$

$$\rho_n(P) = \gamma_n(P)\,\theta_{AB}^{CP}(\zeta_n). \tag{4.9}$$

It is not difficult to verify that if one chooses the functions α_n, β_n, γ_n as follows:

$$\alpha_n(P) = \alpha_*\mu^{n^2/2}\exp\{n\varphi_{DC}(P)\} \tag{4.10}$$

$$\beta_n(P) = q_*\varepsilon^n\exp\{\varphi_{AC}(P)\}\alpha_n(P) \tag{4.11}$$

$$\gamma_n(P) = r_*\varepsilon^{-n}\exp\{-\varphi_{AC}(P)\}\alpha_n(P) \tag{4.12}$$

where the functions φ_{QR} are defined in the vicinity of the point B by

$$\exp\{\varphi_{QR}(P) - \varphi_{QR}(B)\} = \frac{\hat\theta_Q^P}{\hat\theta_R^P}\,\frac{\hat\theta_R^B}{\hat\theta_Q^B} \tag{4.13}$$

the constant μ is given by

$$\mu = \frac{\left(\hat\theta_C^A\right)^2}{\hat\theta_D^A\,\hat\theta_C^B} \tag{4.14}$$

and α_*, q_*, r_* and ε are arbitrary constants satisfying

$$q_* r_* = -\frac{\left(\hat\theta_B^A\right)^2}{\hat\theta_D^A\,\hat\theta_C^B} \tag{4.15}$$

then (4.3)–(4.5) can be rewritten in terms of the functions $\sigma_n(P)$, $\rho_n(P)$ and $\tau_n(P)$ as

$$\tau_n(B)\,\sigma_n(P) - \sigma_n(B)\,\tau_n(P) = K(P)\,\tau_{n-1}(B)\,\sigma_{n+1}(P) \tag{4.16}$$

$$\rho_n(B)\,\tau_n(P) - \tau_n(B)\,\rho_n(P) = K(P)\,\rho_{n-1}(B)\,\tau_{n+1}(P) \tag{4.17}$$

$$\tau_n(B)\,\tau_n(P) - \rho_n(B)\,\sigma_n(P) = \tau_{n-1}(B)\,\tau_{n+1}(P) \tag{4.18}$$

where

$$K(P) = \frac{1}{\varepsilon}\,\frac{\hat\theta_D^A}{\hat\theta_C^B}\,\frac{\hat\theta_B^P\,\hat\theta_C^P}{\hat\theta_A^P\,\hat\theta_D^P}. \tag{4.19}$$

Thus we have presented Fay's identities in the bilinear form similar to (2.18)–(2.20). What I have to do now is to introduce a z dependence in such a way that a shift over the Riemann surface from point B to a point P (which correspond to the points 0 and ξ of the complex plane) can be taken into account by the simultaneous shifts $z_m \to z_m + \mathrm{i}\xi^m/m$:

$$f_n(B) = f_n(z) = f_n(z_1, z_2, z_3, \ldots), \tag{4.20}$$

$$f_n(P) = f_n(z + \mathrm{i}[\xi]) = f_n(z_1 + \mathrm{i}\xi, z_2 + \mathrm{i}\xi^2/2, z_3 + \mathrm{i}\xi^3/3, \ldots). \tag{4.21}$$

(I hope that the usage of the same letters for functional dependence on both the point of the Riemann surface and the ALH variables z_m will not lead to confusion.) In other words, I want to introduce functions $\boldsymbol{\zeta}(z_1, z_2, \ldots)$ and $\varphi_{QR}(z_1, z_2, \ldots)$ such that

$$\boldsymbol{\zeta}(z + \mathrm{i}[\xi]) - \boldsymbol{\zeta}(z) = \int_B^P \boldsymbol{\omega} \tag{4.22}$$

and

$$\varphi_{QR}(z + \mathrm{i}[\xi]) - \varphi_{QR}(z) = \varphi_{QR}(P) - \varphi_{QR}(B). \tag{4.23}$$

This can be done as follows. In the neighbourhood of the point B (which is a preimage of the point $\xi = 0$ of the complex plane) the components of the integral in (4.22) can be presented in terms of the ξ-coordinate as

$$\int_B^P \omega_k = W_k(\xi) = -\sum_{l=1}^{g} C_{k,l} \int_0^\xi \frac{x^{l-1}\,\mathrm{d}x}{\sqrt{\mathcal{P}_{2g+2}(x)}} \tag{4.24}$$

where the sign of the square root is fixed by $\sqrt{1} = 1$. Hence, taking $\boldsymbol{\zeta}$ to be a linear function of the coordinates z_m,

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(z) = \sum_{m=1}^{\infty} \boldsymbol{\zeta}^{(m)}\, z_m \tag{4.25}$$

one can conclude that to satisfy (4.22) the vectors $\boldsymbol{\zeta}^{(m)}$ should be defined as the coefficients of the series

$$\sum_{m=1}^{\infty} \boldsymbol{\zeta}^{(m)}\, \xi^m = -\mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi}\, \boldsymbol{W}(\xi) \tag{4.26}$$

(here $\boldsymbol{W}$ is the vector with the components W_k). Using (4.24) one can rewrite (4.26) as

$$\sum_{m=1}^{\infty} \zeta_k^{(m)}\, \xi^m = \mathrm{i} \sum_{l=1}^{g} C_{k,l} \frac{\xi^l}{\sqrt{\mathcal{P}_{2g+2}(\xi)}} \tag{4.27}$$

(the right-hand side of this equation should be understood as a power series in ξ).

In a similar way one can tackle equation (4.23) and to derive the following result: $\varphi_{QR}(z)$ is the linear function,

$$\varphi_{QR}(z) = \sum_{m=1}^{\infty} \varphi_{QR}^{(m)}\, z_m \tag{4.28}$$

with the coefficients $\varphi^{(m)}_{QR}$ being defined by

$$\sum_{m=1}^{\infty} \varphi^{(m)}_{QR}\,\xi^m = -\mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \frac{\theta[\boldsymbol{\delta}]\left(\int_Q^B \boldsymbol{\omega} + \boldsymbol{W}(\xi)\right)}{\theta[\boldsymbol{\delta}]\left(\int_R^B \boldsymbol{\omega} + \boldsymbol{W}(\xi)\right)} \tag{4.29}$$

(the right-hand side is again a power series in ξ).

Thus one can write the following expressions for the tau-functions:

$$\tau_n(z) = \alpha_* \mu^{n^2/2} \exp\{n\varphi_{DC}(z)\}\,\theta(\boldsymbol{\zeta}_n(z)) \tag{4.30}$$

$$\sigma_n(z) = q_* \varepsilon^n \mu^{n^2/2} \exp\{n\varphi_{DC}(z) + \varphi_{AC}(z)\}\,\theta\left(\boldsymbol{\zeta}_n(z) - \int_A^C \boldsymbol{\omega}\right) \tag{4.31}$$

$$\rho_n(z) = r_* \varepsilon^{-n} \mu^{n^2/2} \exp\{n\varphi_{DC}(z) - \varphi_{AC}(z)\}\,\theta\left(\boldsymbol{\zeta}_n(z) + \int_A^C \boldsymbol{\omega}\right). \tag{4.32}$$

At last, we have to rewrite the function $K(P)$ from the right-hand side of (4.16)–(4.18). This is the first time, since Fay's identity has been written down, that we need some facts from the theory of the Riemann surfaces—until now everything has been done by simple algebra. Consider the function

$$f(P) = \frac{\hat{\theta}^P_B\,\hat{\theta}^P_C}{\hat{\theta}^P_A\,\hat{\theta}^P_D}. \tag{4.33}$$

This is a single-valued (due to condition (4.2)) function which possesses zeros at the points B, C and poles at A, D. Remembering that B, C correspond to $\xi = 0$, and A, D to $\xi = \infty$, one can easily obtain one function with the same divisor, $B + C - A - D$, namely, the projection $\pi(P)$ discussed above (see the paragraph before (4.2)). The quotient $\pi(P)/f(P)$ has no poles (and zeros as well) on X, hence it is a constant,

$$f(P) = C\xi \qquad \text{for} \quad P = (s, \xi). \tag{4.34}$$

Thus, if we take

$$\varepsilon = C \frac{\hat{\theta}^A_D}{\hat{\theta}^B_C} \tag{4.35}$$

then

$$K(P) = \xi \tag{4.36}$$

and relations (4.16)–(4.18) become (2.18)–(2.20), or in other words, the functions defined by (4.30)–(4.32) solve equations (2.18)–(2.20).

Until now we were operating in a neighbourhood of the point B and obtained solutions of equations (2.18)–(2.20), and hence of (2.9) and (2.10), i.e. solved the 'positive' part of the ALH. To take into account the 'negative' equations (2.11) and (2.12), or (2.21)–(2.23), one can proceed in the similar way, but this time considering flows near another distinguished point, D, which is a preimage of the point $\xi = \infty$. It can be shown that functions τ_n, σ_n and ρ_n given by (4.30)–(4.32) will solve (2.21)–(2.23) provided we introduce the $\bar{z}$ dependence by replacing

$$\boldsymbol{\zeta}(z) \to \boldsymbol{\zeta}(z, \bar{z}) \tag{4.37}$$

$$\varphi_{DC}(z) \to \varphi_{DC}(z) + \bar{\varphi}_{BA}(\bar{z}) \tag{4.38}$$

$$\varphi_{AC}(z) \to \varphi_{AC}(z) + \bar{\varphi}_{CA}(\bar{z}) \tag{4.39}$$

(the overbar *does not* mean complex conjugation!) where

$$\boldsymbol{\zeta}(z, \bar{z} + \mathrm{i}[\xi^{-1}]) - \boldsymbol{\zeta}(z, \bar{z}) = \overline{\boldsymbol{W}}(\xi^{-1}) = \int_D^P \boldsymbol{\omega} \tag{4.40}$$

and

$$\bar{\varphi}_{QR}(\bar{z} + \mathrm{i}[\xi^{-1}]) - \bar{\varphi}_{QR}(\bar{z}) = \ln \frac{\hat{\theta}_Q^P}{\hat{\theta}_R^P} \frac{\hat{\theta}_R^D}{\hat{\theta}_Q^D}. \tag{4.41}$$

Thus, now we have all that is necessary to formulate the main result of this paper. The finite-genus solutions of the ALH can be presented as

$$q_n(z, \bar{z}) = q_* \varepsilon^n \exp\{\varphi(z, \bar{z})\} \frac{\theta(\boldsymbol{\zeta}(z, \bar{z}) + n\boldsymbol{U} - \boldsymbol{V})}{\theta(\boldsymbol{\zeta}(z, \bar{z}) + n\boldsymbol{U})} \tag{4.42}$$

$$r_n(z, \bar{z}) = r_* \varepsilon^{-n} \exp\{-\varphi(z, \bar{z})\} \frac{\theta(\boldsymbol{\zeta}(z, \bar{z}) + n\boldsymbol{U} + \boldsymbol{V})}{\theta(\boldsymbol{\zeta}(z, \bar{z}) + n\boldsymbol{U})} \tag{4.43}$$

where

$$\boldsymbol{U} = \int_A^B \boldsymbol{\omega} \qquad \boldsymbol{V} = \int_A^C \boldsymbol{\omega}. \tag{4.44}$$

The functions $\boldsymbol{\zeta}(z, \bar{z})$ and $\varphi(z, \bar{z})$ are given by

$$\boldsymbol{\zeta}(z, \bar{z}) = \sum_{m=1}^{\infty} \left(\boldsymbol{\zeta}^{(m)} z_m + \bar{\boldsymbol{\zeta}}^{(m)} \bar{z}_m \right) + \text{constant} \tag{4.45}$$

$$\varphi(z, \bar{z}) = \sum_{m=1}^{\infty} \left(\varphi^{(m)} z_m + \bar{\varphi}^{(m)} \bar{z}_m \right) + \text{constant} \tag{4.46}$$

where the constants $\boldsymbol{\zeta}^{(m)}, \bar{\boldsymbol{\zeta}}^{(m)}$ and $\varphi^{(m)}, \bar{\varphi}^{(m)}$ are defined as coefficients of the series

$$\sum_{m=1}^{\infty} \zeta_k^{(m)} \xi^m = \mathrm{i} \sum_{l=1}^{g} C_{k,l} \frac{\xi^l}{\sqrt{\mathcal{P}_{2g+2}(\xi)}} \tag{4.47}$$

$$\sum_{m=1}^{\infty} \bar{\zeta}_k^{(m)} \xi^{-m} = -\mathrm{i} \sum_{l=1}^{g} C_{k,g+1-l} \frac{\xi^{-l}}{\sqrt{\overline{\mathcal{P}}_{2g+2}(1/\xi)}} \tag{4.48}$$

with

$$\overline{\mathcal{P}}_{2g+2}(\xi) = \xi^{2g+2}(\xi) \mathcal{P}_{2g+2}(1/\xi) \tag{4.49}$$

and

$$\sum_{m=1}^{\infty} \varphi^{(m)} \xi^m = \mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \frac{\theta[\boldsymbol{\delta}](\boldsymbol{U} - \boldsymbol{V} + \boldsymbol{W}(\xi))}{\theta[\boldsymbol{\delta}](\boldsymbol{U} + \boldsymbol{W}(\xi))} \tag{4.50}$$

$$\sum_{m=1}^{\infty} \bar{\varphi}^{(m)} \xi^{-m} = \mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \frac{\theta[\boldsymbol{\delta}](\boldsymbol{U} + \overline{\boldsymbol{W}}(1/\xi))}{\theta[\boldsymbol{\delta}](\boldsymbol{U} + \boldsymbol{V} + \overline{\boldsymbol{W}}(1/\xi))}. \tag{4.51}$$

The constant ε is given by (4.35) and q_*, r_* are arbitrary constants related by (4.15).

The 'real' tau-function τ_n can be written as

$$\tau_n(z, \bar{z}) = \alpha_* \mu^{n^2/2} \exp\{n\psi(z, \bar{z})\} \theta(\boldsymbol{\zeta}(z, \bar{z}) + n\boldsymbol{U}) \tag{4.52}$$

where the constant μ is given by (4.14),

$$\psi(z,\bar{z}) = \sum_{m=1}^{\infty}\left(\psi^{(m)} z_m + \bar{\psi}^{(m)} \bar{z}_m\right) + \text{constant} \tag{4.53}$$

and

$$\sum_{m=1}^{\infty} \psi^{(m)} \xi^m = \mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \frac{\theta[\boldsymbol{\delta}](\boldsymbol{U} - \boldsymbol{V} + \boldsymbol{W}(\xi))}{\theta[\boldsymbol{\delta}](-\boldsymbol{V} + \boldsymbol{W}(\xi))} \tag{4.54}$$

$$\sum_{m=1}^{\infty} \bar{\psi}^{(m)} \xi^{-m} = \mathrm{i}\xi \frac{\mathrm{d}}{\mathrm{d}\xi} \ln \frac{\theta[\boldsymbol{\delta}](\boldsymbol{V} + \overline{\boldsymbol{W}}(1/\xi))}{\theta[\boldsymbol{\delta}](\boldsymbol{U} + \boldsymbol{V} + \overline{\boldsymbol{W}}(1/\xi))}. \tag{4.55}$$

5. Discussion

In this paper we have obtained the finite-genus solutions for the ALH. The results can also be used to derive the finite-genus solutions for other integrable hierarchies, which can be 'embedded' into the ALH (see [10–13]. So, for example, the functions

$$Q = \frac{\sigma_1}{\tau_0} = Q_* \exp\{\psi(z,\bar{z}) + \varphi(z,\bar{z})\} \frac{\theta(\boldsymbol{\zeta}(z,\bar{z}) - \widetilde{\boldsymbol{V}})}{\theta(\boldsymbol{\zeta}(z,\bar{z}))} \tag{5.1}$$

$$R = \frac{\rho_{-1}}{\tau_0} = R_* \exp\{-\psi(z,\bar{z}) - \varphi(z,\bar{z})\} \frac{\theta(\boldsymbol{\zeta}(z,\bar{z}) + \widetilde{\boldsymbol{V}})}{\theta(\boldsymbol{\zeta}(z,\bar{z}))} \tag{5.2}$$

where

$$\widetilde{\boldsymbol{V}} = \boldsymbol{V} - \boldsymbol{U} = \int_B^C \boldsymbol{\omega} \tag{5.3}$$

and constants Q_*, R_* are related by

$$Q_* R_* = -\left[\varepsilon \frac{\hat{\theta}_B^A \, \hat{\theta}_C^A}{\hat{\theta}_D^A \, \hat{\theta}_C^B}\right]^2 \tag{5.4}$$

solve the nonlinear Schrödinger equation

$$\mathrm{i}\partial_2 Q + \partial_{11} Q + 2Q^2 R = 0 \tag{5.5}$$

$$-\mathrm{i}\partial_2 R + \partial_{11} R + 2QR^2 = 0 \tag{5.6}$$

where $\partial_m = \partial/\partial z_m$, as well as all higher equations of the hierarchy (see [10]).

The quantities

$$p_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = \mu \frac{\theta(\boldsymbol{\zeta}_T(x,\bar{x}) + (n-1)\boldsymbol{U})\theta(\boldsymbol{\zeta}_T(x,\bar{x}) + (n+1)\boldsymbol{U})}{\theta^2(\boldsymbol{\zeta}_T(x,\bar{x}) + n\boldsymbol{U})} \tag{5.7}$$

where

$$\boldsymbol{\zeta}_T(x,\bar{x}) = \boldsymbol{\zeta}^{(1)} x + \bar{\boldsymbol{\zeta}}^{(1)} \bar{x} + \text{constant} \tag{5.8}$$

solve the 2D Toda lattice equation

$$\frac{\partial^2}{\partial x \partial \bar{x}} \ln p_n = p_{n-1} - 2p_n + p_{n+1}. \tag{5.9}$$

In [12] the relations between the ALH and the Davey–Stewartson equation (together with the Ishimori model) have been derived. One can find there expressions for the corresponding finite-genus solutions.

The last example stems from the fact that for any fixed n the quantity

$$u = r_{n-1} p_n q_{n+1} = \frac{\rho_{n-1}\sigma_{n+1}}{\tau_n^2} \tag{5.10}$$

solves the Kadomtsev–Petviashvili (KP) equation,

$$\partial_1(4\partial_3 u + \partial_{111} u + 12 u \partial_1 u) = 3\partial_{22} u. \tag{5.11}$$

Thus, the results of the previous section yield the following finite-genus solution for the KP:

$$u = Q_* R_* \frac{\theta(\boldsymbol{\zeta}_{KP}(z_1, z_2, z_3) - \widetilde{\boldsymbol{V}})\,\theta(\boldsymbol{\zeta}_{KP}(z_1, z_2, z_3) + \widetilde{\boldsymbol{V}})}{\theta^2(\boldsymbol{\zeta}_{KP}(z_1, z_2, z_3))} \tag{5.12}$$

$$\boldsymbol{\zeta}_{KP}(z_1, z_2, z_3) = \boldsymbol{\zeta}^{(1)} z_1 + \boldsymbol{\zeta}^{(2)} z_2 + \boldsymbol{\zeta}^{(3)} z_3 + \text{constant}. \tag{5.13}$$

Here I set $n = 0$ in (5.10) and omitted the $\bar{z}_m$ dependence for all m as well as the dependence on z_m for $m > 3$. This solution differs from the already known one which corresponds to an odd-order polynomial $\mathcal{P}_{2g+1}$ and, which is crucial, has been obtained by considering flows near the infinity ($\pi(P) = \infty$) which in this case is a ramification (Weierstrass) point (the point $\xi = \infty$ has only one preimage on the Riemann surface). I cannot at present discuss solution (5.12) in detail. For example, I do not know whether it is possible to obtain from (5.12) any non-trivial *real* solutions. In any case, solution (5.12) seems to be interesting and worthy of subsequent studies.

To conclude, I want to point out the main differences between the approach of this paper and ones used earlier [2–6]. In the IST-based methods the hyperelliptic curves appear in the analysis of the spectral data of the scattering problem (2.1), while the dependence on the coordinates is derived from the system (2.2). Here we did not use the zero-curvature representation (2.1) and (2.2) explicitly (though it is surely hidden in the functional equations (2.9)–(2.12)). We started with an almost arbitrary polynomial $\mathcal{P}_{2g+2}$ (the fact that it is related to the transfer matrix of the scattering problem (2.1) was not crucial for our consideration) and obtained the z_j, $\bar{z}_j$ dependence *directly* from the equations of the ALH (and not from the corresponding equations for the transfer matrix T_n).

As to the algebro-geometrical method, the main distinguishing point is that the approach of the work [6] (and analogous works devoted to other integrable equations) is, so to say 'global', while ours is 'local'. The authors of [6] used the Baker–Akhiezer function and other structures defined for the whole Riemann surface X. At the same time we did not use globally defined objects: each time we introduced some functions depending on the point P of the Riemann surface X it was understood that it is defined in some vicinity of some distinguished point (B or D). We even did not discuss the question of whether our functions, say $\varphi_{QR}(P)$, are well defined, or single-valued, for all P, $P \in X$. All we needed was the Taylor expansions, say (4.28), hence for our purposes it was enough that our functions exist locally, for P belonging to some (arbitrary small) neighbourhood of the point B (or D).

Lastly, I would like to note that the idea of applying Fay's identity to differential equations is far from new. For example, in the book [8] one can find few examples of how to demonstrate that θ-functions of some arguments solve the KP, KdV, sine–Gordon equations. However, in these examples the main question is how some *combinations* of differential operators (flows) act on a θ-function. To my knowledge the problem of action of these operators taken *separately* has not been considered before. Now we know a partial answer: the flows near a regular (not Weierstrass) point of a Riemann surface can be described by means of the equations of the ALH. Such an appearance of the ALH seems to be new and rather interesting. Combined with the results of the works [9–13] it can be viewed as one more point indicating the 'universality' of this hierarchy.

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